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On maximal immediate extensions of valued fields

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We study the maximal immediate extensions of valued fields whose residue fields are perfect and whose value groups are divisible by the residue characteristic if it is positive. In the case where there is such an extension which has finite transcendence degree we derive strong properties of the field and the extension and show that the maximal immediate extension is unique up to isomorphism, although these fields need not be Kaplansky fields. If the maximal immediate extension is an algebraic extension, we show that it is equal to the perfect hull and the completion of the field.

1 Introduction

In this paper, we denote a valued field by (K, v), its value group by vK, and its residue field by Kv. When we talk of a valued field extension $(L|K, v)$ we mean that (L, v) is a valued field, $L|K$ a field extension, and K is endowed with the restriction of v. For the basic facts about valued fields, we refer the reader to [5, 6, 14, 17, 20, 21].

A **henselian field** is a valued field which satisfies Hensel's Lemma, or equivalently, admits a unique extension of its valuation to its algebraic closure. Note that every algebraic extension of a henselian field is again henselian, with respect to the unique extension of the valuation. A **henselization** of a valued field (K, v) is an algebraic extension which is henselian and minimal in the sense that it can be embedded in every other henselian extension field of (K, v). Henselizations exist for every valued field (K, v), and they are unique up to valuation preserving isomorphism over K. Therefore, we will speak of "the henselization of (K, v)" and denote it by (K^h, v).

An extension $(L|K, v)$ is called **immediate** if the canonical embeddings of vK in vL and of Kv in Lv are onto, in other words, value group and residue field remain unchanged. Henselizations are immediate separable-algebraic extensions. A valued field is called **maximal** if it does not admit any nontrivial immediate extensions. It follows that a maximal immediate extension of a valued field is a maximal field. It was shown by W. Krull in [9] that every valued field (K, v) admits a maximal immediate extension (M, v) (the proof was later simplified by K. A. H. Gravett in [7]). However, the maximal immediate extension M does not need to be unique up to isomorphism. This was shown by I. Kaplansky in [8]. He proved also that under a certain condition, called "hypothesis A", uniqueness holds (see below). A valued field (K, v) satisfying hypothesis A is called a **Kaplansky field**. By Theorem 1 of [22], hypothesis A is equivalent to the conjunction of the following three conditions, where p denotes the characteristic char Kv of the residue field:

(K1) if $p > 0$ then the value group vK is p-divisible,

(K2) the residue field Kv is perfect,

(K3) the residue field Kv admits no finite separable extension of degree divisible by p.

A more elementary proof for the equivalence was later given by Kaplansky himself, based on an idea of D. Leep, and is documented in [10]. Note that conditions (K2) and (K3) can be combined into the condition that the residue field Kv admits no finite extensions of degree divisible by its characteristic. But splitting this up has a purpose; see the paper [15] which presents an alternative proof of the following theorem:

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Theorem 1.1 (Kaplansky, Theorem 5 of [8]) *If (K, v) is a Kaplansky field, then the maximal immediate extension of (K, v) is unique up to valuation preserving isomorphism over K.*

Note further that every valued field of residue characteristic 0 is a Kaplansky field.

The question arises whether the converse is also true: if a valued field does not satisfy hypothesis A, does the uniqueness then necessarily fail? The answer is no, as the next theorem will show the existence of valued fields that are not maximal and not Kaplansky fields but admit proper unique maximal immediate extensions:

Theorem 1.2 *Assume that (K, v) satisfies* **(K1)** *and* **(K2)**. *If (K, v) admits a maximal immediate extension of finite transcendence degree, then all maximal immediate extensions of (K, v) are isomorphic over K, as valued fields. If (K, v) admits an immediate extension of infinite transcendence degree, then all maximal immediate extensions of (K, v) are of infinite transcendence degree over K.*

To produce examples of the above mentioned fields, take any valued field (K_0, v) that is not maximal and satisfies **(K1)** and **(K2)**, but not **(K3)**. Take any maximal immediate extension (M, v) of (K_0, v) and denote the transcendence basis of $M|K_0$ by T. Choose any finite nonempty subset T_0 of T. Then the henselization of $(K_0(T \setminus T_0), v)$ does not satisfy **(K3)**, but by Theorem 1.2, its proper maximal immediate extension is unique up to valuation preserving isomorphism.

We are now going to study the situation described in the previous theorem in more detail. First, we introduce some further helpful notions. An algebraic extension $(L|K, v)$ of a henselian field (K, v) is called **tame** if every finite subextension $F|K$ of $L|K$ satisfies the following conditions:

(T1) if char $Kv = p > 0$, then the ramification index $(vF : vK)$ is prime to p,

(T2) the residue field extension $Fv|Kv$ is separable,

(T3) $(F|K, v)$ is a **defectless extension**, i.e., $[F : K] = (vF : vK) \cdot [Fv : Kv]$.

Assume that (K, v) is a henselian field with char$Kv = 0$. Then the first two conditions of the above definition are trivially satisfied and the third one follows immediately from the Lemma of Ostrowski (see (2) below). Hence every algebraic extension of such a field is tame.

Denote by $\tilde{K}$ the algebraic closure and by K^{sep} the separable-algebraic closure of K. A henselian valued field (K, v) is said to be a **tame field** if $(\tilde{K}|K, v)$ is a tame extension, and a **separably tame field** if $(K^{\mathrm{sep}}|K, v)$ is a tame extension. From the definition of a tame extension it follows that (K, v) is tame if and only if

(TF1) if char $Kv = p > 0$, then the value group vK is p-divisible,

(TF2) the residue field Kv is perfect,

(TF3) (K, v) is a **defectless field**, i.e., each of its finite extensions is defectless.

Note that by **(TF1)** and **(TF2)**, the perfect hull of a tame field is an immediate extension, and by **(TF3)**, this extension must be trivial. This shows that every tame field is perfect. To obtain a characterization of separably tame fields one only has to restrict condition **(TF3)** to finite separable extensions.

Theorem 1.3 *Assume that (K, v) satisfies* **(K1)** *and* **(K2)**. *Assume further that (M, v) is nontrivially valued and maximal, and a (not necessarily immediate) extension of (K, v) of finite transcendence degree. Take $L|K$ to be the maximal separable-algebraic subextension of $M|K$. Then L contains a henselization K^h of (K, v), and we have:*

a) K^h is a separably tame field,

b) $L|K^h$ is a finite tame extension,

c) the perfect hull of K is contained in the completion of K, and the same holds for K^h in place of K,

d) vM/vK and $Mv|Kv$ are finite.

If in addition, $M|K$ is algebraic, then we also have:

e) M is equal to the perfect hull of L and to the completion of L,

f) the perfect hull of K^h is equal to the completion of K^h and is the unique maximal immediate extension of K^h.

The question under which additional conditions the converse of Theorem 1.1 holds was studied in [19] and in [15]. Propositions 8.2 and 8.5 in the latter paper provide such an additional condition under which a henselian field satisfying **(K1)** and **(K2)** but violating **(K3)** will always have two nonisomorphic maximal immediate extensions. But this condition is violated by henselian fields that satisfy the assumption of the previous theorem since by part a) of the theorem they are separably tame.

The following corollary is a byproduct of the proof of Theorem 1.3.

Corollary 1.4 *Take a valued field (K, v) which satisfies* **(K1)** *and* **(K2)** *and consider the following assertions.*

(i) The perfect hull of K is not contained in the completion of K.
(ii) There is a finite separable-algebraic extension $(F|K, v)$ such that the valuation v extends in a unique way from K to F and $(F|K, v)$ is not defectless.
(iii) Every maximal immediate extension of (K, v) is of infinite transcendence degree over K.

Then (i) implies (ii) and (ii) implies (iii).

Note that a field with the trivial valuation is maximal. Together with Theorem 1.3 this yields the following:

Corollary 1.5 *Take a valued field (K, v) which satisfies* **(K1)** *and* **(K2)***, and assume that (M, v) is a maximal immediate extension of (K, v) such that $M|K$ is algebraic. For K^h the henselization of (K, v) inside of (M, v), we have:*

a) if $\operatorname{char} K = 0$*, then $M = K^h$, so (K^h, v) is maximal;*

b) otherwise, M is equal to the perfect hull of K^h and to the completion of K^h.

Note that if in the situation of the above corollary the extension $M|K$ is separable-algebraic, then regardless of the characteristic of K we obtain that (K^h, v) is a maximal field. The next corollary shows that this holds also if we do not require $M|K$ to be immediate.

Corollary 1.6 *Take a valued field (K, v) which satisfies* **(K1)** *and* **(K2)***, with v nontrivial on K, and a separable-algebraic extension M of K. If (M, v) is maximal, then the henselization (K^h, v) of (K, v) inside of (M, v) is maximal and a tame field, and $(M|K^h, v)$ is a finite tame extension.*

The proofs of Theorems 1.2 and 1.3 and of Corollaries 1.4 and 1.6 will be given in Section 3.

The conclusion of Corollary 1.6 remains true if we replace "separable-algebraic extension" by "finite extension". A valued field (K, v) is called **maximal-by-finite** if it is not maximal, but a finite extension (M, v) of (K, v) is a maximal field. First we discuss the case of (K, v) being henselian. The following theorem will be proved in Section 2.3.

Theorem 1.7 *Take a henselian valued field (K, v) which satisfies* **(K1)** *and* **(K2)***, and a finite extension (M, v) of (K, v). If (M, v) is maximal, then (K, v) is maximal and a tame field, and $(M|K, v)$ is a tame extension. Hence, a henselian valued field which satisfies* **(K1)** *and* **(K2)** *cannot be maximal-by-finite.*

This theorem constitutes a partial answer, for the case that (K, v) satisfies **(K1)** and **(K2)**, to a question we have been asked by K. Struyve and K. Kedlaya: *Let K be a field of characteristic $p > 0$ complete for a real valuation. Suppose that there exists an Artin-Schreier extension L of K which is maximal. Does this imply that K is maximal?*

Maximal-by-finite fields are of importance in connection with decomposition problems for modules over valuation domains. For this purpose, the characterization of non-henselian maximal-by-finite fields has already been given by P. Vamos in [18]. His result can be formulated as follows:

Theorem 1.8 *Suppose that (K, v) is maximal-by-finite, but not henselian. Then K is formally real, Kv is algebraically closed (and hence v is not compatible with any ordering on K), v admits two distinct immediate extensions to $K(\sqrt{-1})$ and with both of them, $K(\sqrt{-1})$ is maximal. Exactly one of the following two cases holds:*

i) K is real closed, $K(\sqrt{-1})$ is algebraically closed and the completion of (K, v).
ii) v is a composition $v = w \circ \overline{w}$ with both w and $\overline{w}$ nontrivial, (K, w) is maximal and case i) holds for $(Kw, \overline{w})$; in this case, (K, v) is complete.

These two theorems leave open the case of henselian fields that violate **(K1)** or **(K2)**. Not much seems to be known in this case. M. Nagata ([16, Appendix, Example (E3.1), pp. 206-207]) gave an example of a discrete valued field which is not maximal but has a finite purely inseparable maximal immediate extension. This example has been generalized in Theorem 1.5 of [4], which shows that non-discrete valued fields with the same property exist. All fields in these examples have infinite p-degree, and this is in fact necessary, according to the next result which will be proved in Section 2.3.

Proposition 1.9 *If the valued field (K,v) admits a proper finite purely inseparable maximal immediate extension, then the p-degree of K is infinite. More generally, if the p-degree of K is finite and M is a finite purely inseparable extension of K such that (M,v) is maximal, then (K,v) is maximal.*

But there appear to be no corresponding results in the literature for finite not purely inseparable maximal immediate extensions of henselian fields that violate **(K1)** or **(K2)**. However, there are related results that show the influence of the p-degree on the behaviour of maximal immediate extensions: see Theorems 1.2, 1.5 and 1.6 of [4].

Open question: What can be said about a valued field that admits a maximal immediate extension of finite transcendence degree but violates **(K1)** or **(K2)**?

This paper is in part based on the thesis [2] of the first author.

2 Preliminaries

By char K we denote the characteristic of K. In slight abuse of notation, we will denote the perfect hull of a field K by K^{1/p^∞} even if in the respective context, p denotes a prime other than char K. If char $K = 0$, then $K^{1/p^\infty} = K$ even if K is valued with char $Kv = p > 0$.

2.1 Tame extensions and defect extensions

Take a finite extension $(L|K,v)$ of valued fields. If $v_1 = v, \ldots, v_g$ are the distinct extensions of the valuation v of K to the field L, then $L|K$ satisfies the **fundamental inequality** (cf. Corollary 17.5 of [5]):

$$[L:K] \;\geq\; \sum_{i=1}^{g} (v_iL : vK)[Lv_i : Kv]. \tag{1}$$

If in addition the extension of v from K to L is unique, then the **Lemma of Ostrowski** (see [21], Chapter VI, §12, Corollary to Theorem 25) says that

$$[L:K] \;=\; p^\nu (vL : vK)[Lv : Kv] \tag{2}$$

for a nonnegative integer ν and p the **characteristic exponent** of Kv, that is, $p = \mathrm{char}Kv$ if it is positive and $p = 1$ otherwise. The factor $d(L|K,v) := p^\nu$ is called the **defect** of the extension $(L|K,v)$. If it is nontrivial, that is, if $\nu > 0$, then $(L|K,v)$ is called **a defect extension**. Otherwise, as mentioned already in the introduction, $(L|K,v)$ is called a **defectless extension**.

A useful consequence of inequality (1) is the following fact.

Lemma 2.1 *If L is a finite extension of a valued field (K,v), then the extension of v from K to L is unique if and only if $L|K$ is linearly disjoint from some (equivalently, every) henselization of (K,v).*

Proof. By Corollary 7.48 of [14], we have that

$$[L:K] = \sum_{i=1}^{g} [L^{h(v_i)} : K^{h(v_i)}] = \sum_{i=1}^{g} [L.K^{h(v_i)} : K^{h(v_i)}], \tag{3}$$

where $v_1, \ldots, v_g$ are the distinct extensions of v to L and $L^{h(v_i)}$, $K^{h(v_i)}$ are henselizations of L and K with respect to an extension of v_i to $\tilde{L} = \tilde{K}$. If $L|K$ is not linearly disjoint from $K^{h(v_i)}|K$ for some $i \leq g$, then $[L.K^{h(v_i)} : K^{h(v_i)}] < [L:K]$. Thus $g \geq 2$ and the extension of v from K to L is not unique.

On the other hand, if $L|K$ is linearly disjoint from some henselization K^h of K, then $[L:K] = [L.K^h : K^h]$ and from equation (3) we deduce that v admits a unique extension from K to L. □

Assume now that the extension of the valuation v from K to L is unique. Then inequality (1) is of the form

$$[L:K] \;\geq\; (vL:vK)[Lv:Kv], \tag{4}$$

and the "missing factor" on the right hand side of the inequality is determined by the Lemma of Ostrowski, i.e., equation (2). Fix an extension of v from K to $\tilde{K}$ and denote it again by v. Then from Lemma 2.1 it follows that $[L:K]=[L.K^h:K^h]=[L^h:K^h]$. Since $(K^h|K,v)$ and $(L^h|L,v)$ are immediate extensions, $(vL:vK)[Lv:Kv]=(vL^h:vK^h)[L^hv:K^hv]$. Together with the definition of the defect, this yields that

$$d(L|K,v) \;=\; d(L^h|K^h,v). \tag{5}$$

It follows that if $(L|K,v)$ is a defect extension, then also $(L^h|K^h,v)$ has nontrivial defect.

Take a valued field (K,v), fix an extension of v to K^{sep} and call it again v. The fixed field of the closed subgroup

$$G^r := \{\sigma\in \mathrm{Gal}\,(K^{\mathrm{sep}}|K) \mid v(\sigma a - a) > va \text{ for all } a\in\mathcal{O}_{K^{\mathrm{sep}}}\setminus\{0\}\}$$

of Gal$(K^{\mathrm{sep}}|K)$ (cf. Corollary 20.6 of [5]) is called the **absolute ramification field of** (K,v) and is denoted by $(K,v)^r$ or K^r if v is fixed. Ramification theory states that $K^r=\tilde{K}$ if char $Kv=0$, and $K^{\mathrm{sep}}|K^r$ is a p**-extension** if char $Kv=p>0$, i.e., a Galois extension with a pro-p-group as its Galois group (cf. Lemma 2.7 of [11]). Moreover, for any algebraic extension L of K we have that

$$L^r = L.K^r \tag{6}$$

(cf. Theorem 4.10 of [12]). From Section 4 of [12], equation (4.5), it follows that $K^h\subseteq K^r$. In particular, together with equation (6) it shows that

$$(K^h)^r = K^r. \tag{7}$$

If (K,v) is henselian, then K^r is a Galois extension of K, and it is also the unique maximal tame extension of (K,v) (see Theorem 20.10 of [5] and Proposition 4.1 of [15]). This yields that every tame extension of valued fields is separable-algebraic. Furthermore, we obtain that (K,v) is a tame field if and only if $K^r=\widetilde{K}$.

Lemma 2.2 *Take a normal extension N of a henselian field (K,v) and set $L=N\cap K^r$. Then vN/vL is a p-group, $Nv|Lv$ is purely inseparable and $(L|K,v)$ is a tame extension.*

P r o o f. Since $L|K$ is a subextension of the tame extension $K^r|K$, it is also tame. The assertions on value group and residue field follow from general ramification theory. □

We now turn to a result that will be crucial for the proof of our main theorems. In order to prove it, we need the following lemma.

Lemma 2.3 *The residue field of a henselian valuation on an ordered field has characteristic 0.*

P r o o f. If the residue field of the henselian field (K,v) has characteristic $p>0$, then the reduction of the polynomial X^2-X+p under v is the polynomial X^2-X which has the two distinct roots 0 and 1. Hence by Hensel's Lemma, X^2-X+p splits in K, which is impossible in any ordered field. □

Theorem 2.4 *Assume that $(L|K,v)$ is a finite extension of henselian fields. If (L,v) is a tame field, then also (K,v) is a tame field and the extension $(L|K,v)$ is defectless.*

P r o o f. Since $L^r=L.K^r$, we know that $L^r|K^r$ is a finite extension. Since (L,v) is assumed to be a tame field, we have that L^r is equal to the algebraic closure $\tilde{L}=\tilde{K}$. By Artin-Schreier Theory, K^r is either algebraically closed or real closed. The latter is not possible. Indeed, since v is henselian on K^r, it would have residue characteristic 0 by the previous lemma. But then, (K^r,v) would be a tame field and thus algebraically closed. We conclude that K^r is algebraically closed, showing that (K,v) is a tame field. By definition, it follows that the finite extension $(L|K,v)$ is tame and hence defectless. □

For the conclusion of this section we list two more useful results. The following is Lemma 4.15 of [13]:

Lemma 2.5 *Assume that (L, v) is a tame field and K is a relatively algebraically closed subfield of L. If in addition $Lv|Kv$ is an algebraic extension, then (K, v) is also a tame field.*

One of the ingredients in the proof of the previous lemma is the following fact, which is proved by use of Hensel's Lemma (see, e.g., Lemma 2.4 of [4]).

Lemma 2.6 *Assume (L, v) to be henselian and K to be relatively separable-algebraically closed in L. Then Kv is relatively separable-algebraically closed in Lv. If in addition $Lv|Kv$ is algebraic, then the torsion subgroup of vL/vK is a p-group, where p is the characteristic exponent of Kv.*

2.2 Defect extensions of prime degree

In the study of defect extensions, reduction to the case of defect extensions of prime degree is a crucial tool. This reduction is achieved thanks to the following important property of the absolute ramification field, which is deduced via Galois correspondence from the fact that G^r is a pro-p-group. For the proof, see Lemma 2.9 of [11].

Lemma 2.7 *Let (K, v) be a valued field and take p to be the characteristic exponent of Kv. Then every finite extension of K^r is a tower of normal extensions of degree p. If $L|K$ is a finite extension, then there is already a finite tame extension N of K^h such that $L.N|N$ is such a tower.*

The defect is preserved under liftings through tame extensions (see Proposition 2.8 of [11]):

Proposition 2.8 *Take a henselian field (K, v) and a tame extension N of K. Then for any finite extension $L|K$,*

$$d(L|K, v) = d(L.N|N, v).$$

Take a valued field (K, v) of positive residue characteristic p. Fix an extension of v to K^{sep}. Denote by K^h and K^r the henselization and the absolute ramification field of K with respect to this extension. Take any finite extension $(L|K, v)$ such that the extension of the valuation v from K to L is unique. Then equation (5) together with Proposition 2.8 and equation (7) give that

$$d(L|K, v) = d(L.K^h|K^h, v) = d(L.K^r|K^r, v).$$

On the other hand, Lemma 2.7 shows that $L.K^r|K^r$ is a tower of normal extensions of degree p. Thus, if $L|K$ is separable, then $L.K^r|K^r$ is a tower of Galois extensions of degree p and if $(L|K, v)$ is a defect extension, then so are some of these extensions. This shows that Galois defect extensions of prime degree play a crucial role in the investigation of defect extensions.

If $\mathrm{char}\, K = p$, then every Galois extension of degree p is an **Artin-Schreier extension**, i.e., an extension generated by a root ϑ of a polynomial $X^p - X - a$ with $a \in K$. In this case, ϑ is called an **Artin-Schreier generator** of the extension. On the other hand, if a polynomial $f = X^p - X - a \in K[X]$ has no roots in K, then it is irreducible over the field. If ϑ is a root of f, then the other roots are of the form $\vartheta + 1, \ldots, \vartheta + p - 1$. Hence $K(\vartheta)|K$ is a Galois extension.

Note that an Artin-Schreier extension $(K(\vartheta)|K, v)$ of henselian fields has nontrivial defect if and only if it is immediate. If this holds, we will speak of an Artin-Schreier defect extension. A classification of Artin-Schreier defect extensions (introduced in [11]) distinguishes two types of Artin-Schreier defect extensions, according to their connection with purely inseparable extensions. One type of these extensions can be derived from purely inseparable extensions of degree p by a certain deformation of purely inseparable polynomials into Artin-Schreier polynomials, while the other cannot. The next proposition indicates when such a construction of Artin-Schreier defect extensions is possible; for the proof, see Proposition 4.4 of [11].

Proposition 2.9 *Assume that (K, v) admits an immediate purely inseparable extension of degree p which does not lie in the completion of the field. Then (K, v) admits an Artin-Schreier defect extension.*

Note that if (K, v) satisfies **(K1)** and **(K2)**, then every purely inseparable extension of (K, v) is immediate. Thus if $K^{1/p}$ is not contained in the completion of K, the above proposition yields that (K, v) admits an Artin-Schreier defect extension. However, in the case of p-divisible value group and perfect residue field, we can say much more:

Theorem 2.10 *Take a valued field (K,v) which satisfies* **(K1)** *and* **(K2)**. *If there is a purely inseparable extension of (K,v) which does not lie in the completion of the field, then (K,v) admits an infinite tower of Artin-Schreier defect extensions.*

For the proof see [3, Theorem 1.4].

2.3 Immediate extensions and maximal fields

By (K^c,v) we will denote the completion of a valued field (K,v). It is unique up to valuation preserving isomorphism over K. For some general background about completions of arbitrary valued fields, we refer the reader to Chapter 6 of [14]. The following is Lemma 6.25 of [14].

Lemma 2.11 *Take a finite extension $(L|K,v)$ of valued fields. Then there is a unique extension of v from K^c to $L.K^c$ which coincides with v on L. With this extension, $L^c = L.K^c$.*

The henselization and the completion of a valued field are immediate extensions. This together with Theorem 31.21 of [20] gives the following important properties of maximal fields.

Theorem 2.12 *Every maximal field is henselian, complete and defectless.*

Corollary 2.13 *Take a valued field (K,v) which satisfies* **(K1)** *and* **(K2)**. *Assume that $(M|K,v)$ is an extension such that (M,v) is maximal with vM/vK a torsion group and $Mv|Kv$ algebraic. Then (M,v) is a tame field and hence perfect.*

Proof. Since vK is p-divisible and vM/vK is a torsion group, also vM is p-divisible. Since Kv is perfect and $Mv|Kv$ is algebraic, also Mv is perfect. As every maximal field is defectless by Theorem 2.12, (M,v) is a tame field. □

For the proof of the next result, see Theorem 31.22 of [20].

Theorem 2.14 *Every finite extension of a maximal field is again a maximal field, with respect to the unique extension of the valuation.*

We are going to show that a valued field which satisfies **(K1)** and **(K2)** cannot be maximal-by-finite. To this end, we need the following result, which is Lemma 2.5 of [11].

Lemma 2.15 *Take an immediate extension $(E|K,v)$, an extension of v from E to $\tilde{E}$, and a finite subextension $(L|K,v)$ of $(\tilde{E}|K,v)$ such that the extension of v from K to L is unique and $(L|K,v)$ is defectless. Then $L|K$ is linearly disjoint from $E|K$ and the extension $(E.L|L,v)$ is immediate.*

Corollary 2.16 *Assume that $(L|K,v)$ is a finite extension such that the extension of v from K to L is unique and $(L|K,v)$ is defectless. If (L,v) is maximal, then also (K,v) is maximal.*

Proof. Take any immediate extension $(E|K,v)$. Then by the previous lemma, also $(E.L|L,v)$ is immediate. Since (L,v) is maximal, it must be trivial. But as $L|K$ is linearly disjoint from $E|K$, also $E|K$ must be trivial. This shows that (K,v) is maximal. □

Lemma 2.17 *Assume that (K,v) is a henselian field which is not maximal, and $(M|K,v)$ is a finite extension such that (M,v) is maximal. Then $(M|K,v)$ is a defect extension, and neither (M,v) nor (K,v) is a tame field.*

Proof. Since (K,v) is henselian, the extension of v from K to L is unique. As (K,v) is not maximal, Corollary 2.16 therefore shows that $(M|K,v)$ must be a defect extension. Since every finite extension of a tame field is defectless, this shows that (K,v) cannot be a tame field. Theorem 2.4 now shows that also (M,v) cannot be a tame field. □

Proof of Theorem 1.7. If (M,v) is maximal and a finite extension of (K,v) and the latter satisfies **(K1)** and **(K2)**, then it follows from Corollary 2.13 that (M,v) is a tame field. Since $M|K$ is assumed to be finite, Lemma 2.17 shows that (K,v) must be maximal. Hence it is a defectless field by Theorem 2.12. As it also satisfies **(K1)** and **(K2)** by assumption, it is a tame field, so $(M|K,v)$ is a tame extension. □

For the proof of Proposition 1.9 we have to introduce one more notion. A valued field is called **inseparably defectless** if each of its finite purely inseparable extensions is defectless. (Note that the extension of the valuation is always unique in purely inseparable algebraic extensions.)

Proof of Proposition 1.9. Assume that $M|K$ is a finite purely inseparable extension, (M,v) is maximal and the p-degree of K is finite. Since the p-degree does not change under finite extensions, also the p-degree of M is finite. Now [11, Lemma 3.7] implies that (K,v) is inseparably defectless if and only if (M,v) is. The latter holds because (M,v) is maximal. Hence (K,v) is inseparably defectless, which implies that the extension $(M|K,v)$ is defectless. From Corollary 2.16 we thus obtain that (K,v) is maximal. □

In what follows we will make repeated use of the main theorem of [4]:

Theorem 2.18 *Take a valued field extension $(L|K,v)$ of finite transcendence degree ≥ 0, with v nontrivial on L. Assume that one of the following four cases holds:*

<u>valuation-transcendental case</u>: *vL/vK is not a torsion group, or $Lv|Kv$ is transcendental;*

<u>value-algebraic case</u>: *vL/vK contains elements of arbitrarily high order, or there is a subgroup $\Gamma \subseteq vL$ containing vK such that Γ/vK is an infinite torsion group and the order of each of its elements is prime to the characteristic exponent of Kv;*

<u>residue-algebraic case</u>: *Lv contains elements of arbitrarily high degree over Kv;*

<u>separable-algebraic case</u>: *$L|K$ contains a separable-algebraic subextension $L_0|K$ such that within some henselization of L, the corresponding extension $L_0^h|K^h$ is infinite.*

Then each maximal immediate extension of (L,v) has infinite transcendence degree over L. If the cofinality of vL is countable (which for instance is the case if vL contains an element γ such that $\gamma > vK$), then already the completion of (L,v) has infinite transcendence degree over L.

The next lemma constitutes the main application of this theorem in the present paper.

Lemma 2.19 *Assume that (K,v) is a henselian field which admits an extension (M,v) of finite transcendence degree such that (M,v) is a maximal field and v is nontrivial on M. Then the maximal separable-algebraic subextension of $M|K$ is a finite extension of K.*

P r o o f. Applying Theorem 2.18 with $L=M$, its separable-algebraic case shows that $(M|K,v)$ cannot admit an infinite separable-algebraic subextension, since M is its own maximal immediate extension. □

Corollary 2.20 *Take a separable-algebraic extension $(M|K,v)$ of henselian fields such that (M,v) is a maximal field and v is nontrivial on M. Then the extension is finite and either (K,v) is maximal or $(M|K,v)$ is a nontrivial defect extension.*

P r o o f. The fact that $M|K$ is a finite extension follows directly from the previous lemma.

Suppose that K is not a maximal field. Then it is maximal-by-finite by the assumption on (M,v). Therefore, Lemma 2.17 yields that the extension $(M|K,v)$ has nontrivial defect. □

A valued field (K,v) is called **algebraically maximal** if it does not admit any nontrivial immediate algebraic extension, and **separable-algebraically maximal** if it does not admit any nontrivial immediate separable-algebraic extension. Since henselizations are immediate separable-algebraic extensions, every separable-algebraically maximal field is henselian. In the proof of the following lemma we will make use of the theory of pseudo Cauchy sequences as presented in [8].

Lemma 2.21 *Assume that (K,v) is algebraically maximal. If $(K(x),v)$ is an immediate transcendental extension of (K,v), then $(K(x),v)$ can be embedded over K in every maximal immediate extension of (K,v).*

P r o o f. Since $(K(x),v)$ is an immediate extension of (K,v), Theorem 1 of [8] yields that x is a pseudo limit of a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) without a pseudo limit in (K,v). Since (K,v) admits no nontrivial immediate algebraic extensions, it follows from Theorem 3 of [8] that $(a_\nu)_{\nu<\lambda}$ is of transcendental type. Take any maximal immediate extension (M,w) of (K,v). Then $(a_\nu)_{\nu<\lambda}$ admits a pseudo limit y in (M,w) by Theorem 4 of [8]. From Theorem 2 of [8] we know that sending x to y induces a valuation preserving isomorphism over K from $(K(x),v)$ to $(K(y),w)$, and thus an embedding of $(K(x),v)$ in (M,w). □

3 Proofs of the main results

Throughout this section, we assume that (K, v) is a valued field which satisfies (K1) and (K2).

Proposition 3.1 *Assume that (K, v) admits a maximal immediate extension (M, v) of finite transcendence degree. Then the henselization K^h of (K, v) in (M, v) is relatively separable-algebraically closed in M and the relative algebraic closure of K^h in M is equal to the perfect hull of K^h.*

Proof. If v is trivial on M, then $M = K$, hence the assertion holds.

Suppose now that v is not trivial on M. To simplify notation, we may replace K^h by K (i.e., assume that (K, v) is henselian). As we have already shown in Corollary 2.13, (M, v) is a tame field. Denote by L the relative algebraic closure of K in M. Then Lemma 2.5 shows that also (L, v) is a tame field and hence perfect. It follows that $K^{1/p^\infty} \subseteq L$.

Since v is not trivial on M by assumption, we can use Lemma 2.19 to obtain that the separable-algebraic extension $L|K^{1/p^\infty}$ must be finite. Since (L, v) is tame, by Theorem 2.4 we obtain that $(L|K^{1/p^\infty}, v)$ is defectless. Since it is an immediate extension of henselian fields, it follows that $L = K^{1/p^\infty}$. This proves that K is relatively separable-algebraically closed in M and that $L = K^{1/p^\infty}$. □

Proof of Theorem 1.2. Assume that (M, v) is a maximal immediate extension of (K, v) of finite transcendence degree. Take (N, w) to be another maximal immediate extension of (K, v). It suffices to show that (M, v) can be embedded in (N, w) over K, as a valued field. Indeed, if φ is the embedding, then $(\varphi(M), w)$ is a maximal immediate extension of (K, v). Since the extension $(N|K, w)$ is immediate, also the subextension $(N|\varphi(M), w)$ is immediate and we obtain that $\varphi(M) = N$. Thus, the fields (M, v) and (N, w) are isomorphic over K.

Consider the family of all subextensions $E|K$ of $M|K$ admitting a valuation preserving embedding in N over K. By Zorn's Lemma, there is a maximal such extension $F|K$; denote its embedding by σ. We wish to show that $F = M$.

Since M, N as maximal fields are henselian, M contains the henselization F^h of F and the embedding extends uniquely to an embedding $\tau\colon F^h \to N$. By the maximality of F we have that $F^h = F$, that is, F is henselian.

Take L to be the relative algebraic closure of F in M. Proposition 3.1 yields that $L = F^{1/p^\infty}$. By Corollary 2.13, (N, w) is perfect. Hence, σ can be extended in a unique way to an embedding τ of L in N. From the maximality of σ it follows that $L = F$, that is, F is relatively algebraically closed in M. Again by Corollary 2.13, (M, v) is tame. Therefore, Lemma 2.5 implies that also (F, v) is a tame field.

Since F is relatively algebraically closed in M, either $F = M$ or the extension $M|F$ is transcendental. Suppose the latter holds. We identify (F, v) with its isomorphic image $(\sigma(F), w)$ in (N, w). Take an element $x \in M \setminus F$. Then $(F(x)|F, v)$ is an immediate transcendental extension. By Lemma 2.21, the field $F(x)$ can be embedded in (N, w) over F, a contradiction to the maximality of F. Thus we obtain the required equality $M = F$. This proves the first assertion of our theorem.

The second assertion is an immediate consequence of the first. □

Corollary 3.2 *If (K, v) admits a maximal immediate extension of finite transcendence degree, then its henselization is separable-algebraically maximal.*

Proof. Take an immediate separable-algebraic extension $(E|K^h, v)$. Then (E, v) is contained in some maximal immediate extension (N, w) of (K^h, v). Theorem 1.2 implies that $N|K^h$ is of finite transcendence degree, hence by Proposition 3.1 the field K^h is relatively separable-algebraically closed in N. Thus the extension $E|K^h$ is trivial and consequently, (K^h, v) admits no proper immediate separable-algebraic extensions. □

For the proof of Theorem 1.3, we will need the following lemma.

Lemma 3.3 *Assume that (K, v) is henselian and admits a finite Galois defect extension (E, v). Then every maximal immediate extension of (E, v) is of infinite transcendence degree over E.*

Proof. Set $L = E \cap K^r$. Lemma 2.2 yields that $(L|K, v)$ is a finite tame extension. Furthermore, vE/vL is a p-group and the residue field extension $Ev|Lv$ is purely inseparable. Since vL is p-divisible and Lv is perfect, as this holds already for the value group and the residue field of (K,v), the group vE/vL and the extension $Ev|Lv$ are trivial. Thus $(E|L,v)$ is an immediate extension. From this and Proposition 2.8 it follows that

$$[E:L] \;=\; d(E|L,v) \;=\; d(E|K,v) \;>\; 1.$$

This shows that the immediate separable-algebraic extension $(E|L,v)$ is nontrivial. Applying Corollary 3.2 to the field (L,v) in place of (K,v), we obtain that every maximal immediate extension of (L,v) is of infinite transcendence degree.

Since $(E|L,v)$ is immediate, each maximal immediate extension of (E,v) is also a maximal immediate extension of (L,v); since $E|L$ is algebraic, it must be of infinite transcendence degree too. □

Proof of Theorem 1.3 and Corollary 1.4. Note first that by Lemma 2.19, $L|K^h$ is a finite extension. Take a finite Galois extension E of K^h containing L. Then $E.M|M$ is finite and $(E.M,v)$, where v is the unique extension of the valuation of M to $E.M$, is maximal by Theorem 2.14. Furthermore, from the valuation-transcendental case of Theorem 2.18 it follows that $v(E.M)/vK$ is a torsion group and the residue field extension $(E.M)v|Kv$ is algebraic. Applying Corollary 2.13 to the extension $E.M|K$, we obtain that $(E.M,v)$ is a tame field.

Moreover, since L is relatively-separable algebraically closed in M, the separable-algebraic closure L^{sep} of L is linearly disjoint from M. Since $E|L$ is a subextension of $L^{\mathrm{sep}}|L$, it follows that $L^{\mathrm{sep}} = E^{\mathrm{sep}}$ is linearly disjoint from $E.M$ over E. Hence, E is relatively separable-algebraically closed in $E.M$. Moreover, vE is p-divisible and Ev is perfect. Therefore, it follows from Lemma 2.6 that $(E.M,v)$ is an immediate extension of (E,v). Since $(E.M,v)$ is a maximal field, it is a maximal immediate extension of (E,v).

Suppose that $(E|K,v)$ were a defect extension. Then by Lemma 3.3, every maximal immediate extension of (E,v) would be of infinite transcendence degree. On the other hand, $(E.M,v)$ is a maximal immediate extension of (E,v) of finite transcendence degree, a contradiction.

We thus obtain that every finite Galois extension of K^h containing L is defectless. This implies in particular that every finite separable-algebraic extension of K^h is defectless. Since vK is p-divisible and Kv is perfect, this yields that every finite separable-algebraic extension of K^h is tame. In particular, $(L|K^h,v)$ is a finite tame extension. This proves assertions a) and b).

As we have seen, (K^h,v) admits no separable-algebraic defect extensions. Therefore Theorem 2.10 yields that every purely inseparable extension of (K^h,v) lies in the completion of K^h. It remains to show the same for K in place of K^h; we will prove Corollary 1.4 at the same time.

Assume that assertion (i) of Corollary 1.4 holds. Then we can apply Theorem 2.10 to obtain an Artin-Schreier defect extension $(F|K,v)$. Since both the degree of $F|K$ and the defect are equal to p, the fundamental inequality shows that the extension of v from K to F is unique. We thus have shown that assertion (ii) of Corollary 1.4 holds.

Now assume that assertion (ii) holds. By equation (5), $F^h = F.K^h$ is a finite separable defect extension of (K^h,v). This is a contradiction to what we have already shown about (K^h,v). On the one hand, this contradiction proves that under the assumptions of our theorem, the perfect hull of K is contained in the completion of K, which completes the proof of part c). On the other hand, it shows that if assertion (ii) of Corollary 1.4 holds, then the assertion of our theorem does not hold, which means that assertion (iii) of Corollary 1.4 holds. This completes the proof of Corollary 1.4.

Suppose that vM/vK is infinite. Then by what we have shown in the beginning, it is an infinite torsion group, with all exponents prime to the residue characteristic exponent since vK is p-divisible if $\mathrm{char}\, Kv = p > 0$. Thus, the value-algebraic case of Theorem 2.18 applies. Suppose that $Mv|Kv$ is infinite. Again by what we have shown in the beginning, the extension is algebraic, and it is separable since Kv is perfect. Thus, the residue-algebraic case of Theorem 2.18 applies. In both cases, each maximal immediate extension of (M,v) has infinite transcendence degree over M, which is impossible because (M,v) is itself maximal. This proves assertion d).

Assume from now on that the extension $M|K$ is algebraic. Then $M|L$ is a purely inseparable extension. Moreover, vL is p-divisible and Lv is perfect, as this holds already for vK and Kv. Hence the extension $M|L$ is immediate. Together with the fact that (M,v) is a maximal field, this yields that M is a maximal immediate extension of L. From Proposition 3.1 it follows that M is the perfect hull of L.

Before we show the second assertion of part e), we prove assertion f). Take a maximal immediate extension (N,v) of (K^h,v). As N is henselian, v admits a unique extension to the field $N.L$. Denote this extension again by v. Since the extension $(L|K^h, v)$ is defectless, Lemma 2.15 implies that $(N.L|L, v)$ is an immediate extension. Furthermore, $N.L$ is a finite extension of N. Thus, by Theorem 2.14 we obtain that $N.L$ is a maximal immediate extension of L. By Theorem 1.2 we obtain that $N.L$ is isomorphic to M over L. Thus in particular, $N.L|L$ is an algebraic extension. Hence the same holds for $N|K^h$. This means that (K^h, v) admits a maximal immediate extension algebraic over the field. From Proposition 3.1 it follows that N is equal to the perfect hull of K^h. Moreover, this perfect hull is contained in the completion of K^h, by part c) of our theorem. Since the latter is an immediate extension of K^h, the maximality of N yields that all three fields are equal. As the completion of (K, v) is unique up to isomorphism, this proves assertion f).

It remains to show that the perfect hull of L is equal to the completion of L. Since $L|K^h$ is a finite extension, Lemma 2.11 together with assertion f) yield that

$$L^{1/p^\infty} \;=\; L.(K^h)^{1/p^\infty} \;=\; L.(K^h)^c \;=\; L^c\,.$$

This completes the proof of assertion e). □

Proof of Corollary 1.6. Assume that (M, v) is a maximal field. Then by part b) of Theorem 1.3 we obtain that $(M|K^h, v)$ is a finite tame extension. From Corollary 2.13 it follows that (M, v) is a tame field. Hence, by Theorem 2.4 also (K^h, v) is a tame field. It remains to show that (K^h, v) is a maximal field, but this follows directly from Lemma 2.17. □

Remark 3.4 Assume that (K, v) satisfies the assumptions of part (i) of Corollary 1.4. Then we can give an explicit construction of an immediate extension of (K, v) of infinite transcendence degree. Indeed, if the perfect hull of (K, v) is not contained in the completion of K, then Theorem 2.10 yields that (K, v) admits a separable-algebraic extension $L|K$ which is an infinite tower of Artin-Schreier defect extensions. Now by the separable-algebraic case of Theorem 2.18 we obtain that every maximal immediate extension of (L, v) is of infinite transcendence degree. As every Artin-Schreier defect extension is immediate, we deduce that $(L|K, v)$ is immediate and any maximal immediate extension of L is also a maximal immediate extension of K. Furthermore, the proof of Theorem 2.10 presents a possible construction of the tower of Artin-Schreier defect extensions $(L|K, v)$ (see the proof of Theorem 1.4, [3]). Also the proof of Theorem 2.18 shows how to construct the immediate extension of infinite transcendence degree of (L, v) (cf. Theorem 1.1 of [4]). This gives us a construction of an immediate extension of (K, v) of infinite transcendence degree.

Remark 3.5 Note that if $(L|K, v)$ is a finite separable extension, then the perfect hull of K is contained in the completion K^c of K if and only if the same holds for L. Indeed, if $K^{1/p^\infty} \subseteq K^c$, then by Lemma 2.11 we have that

$$L^{1/p^\infty} = L.K^{1/p^\infty} \subseteq L.K^c = L^c.$$

Conversely, assume that $L^{1/p^\infty} \subseteq L^c$. Then in particular, $K^{1/p^\infty} \subseteq L^c$. By Lemma 2.11 $L^c = L.K^c$. Hence, $L^c|K^c$ is a separable algebraic extension, as $L|K$ is. We thus deduce that $K^{1/p^\infty} \subseteq K^c$.

This shows that we can replace condition (i) of Corollary 1.4 by the equivalent condition:

(i') for some finite separable extension L of K, the perfect hull of L is not contained in the completion of L.

References

[1] N. Bourbaki, Commutative algebra (Paris, 1972).

[2] A. Blaszczok, On the structure of immediate extensions of valued fields, Ph.D. thesis (University of Silesia, 2014).

[3] A. Blaszczok, Infinite towers of Artin-Schreier defect extensions of rational function fields, in: Valuation Theory in Interaction, edited by A. Campillo, F.-V. Kuhlmann, and B. Teissier, EMS Series of Congress Reports (EMS, 2014), 16–54.

[4] A. Blaszczok, and F.-V. Kuhlmann, Algebraic independence of elements in immediate extensions of valued fields, J. Alg. **425**, 179-214 (2015).

[5] O. Endler, Valuation theory (Springer, Berlin, 1972).

[6] A. J. Engler, and A. Prestel, Valued fields, Springer Monographs in Mathematics (Springer-Verlag, Berlin, 2005).

[7] K. A. H. Gravett, Note on a result of Krull, Cambridge Philos. Soc. Proc. **52**, 379 (1956).

[8] I. Kaplansky, Maximal fields with valuations I, Duke Math. Journ. **9**, 303–321 (1942).

[9] W. Krull, Allgemeine Bewertungstheorie, J. reine angew. Math. **167** (1932), 160–196.
[10] F.-V. Kuhlmann, Additive Polynomials and Their Role in the Model Theory of Valued Fields, in: Logic in Tehran, Proceedings of the Workshop and Conference on Logic, Algebra, and Arithmetic, held October 18–22, 2003, edited by A. Enayat, I. Kalantari, and M. Moniri, Lecture Notes in Logic **26** (2006), 160–203.
[11] F.-V. Kuhlmann, A classification of Artin-Schreier defect extensions and a characterization of defectless fields, Illinois J. Math. **54**, 397–448 (2010).
[12] F.-V. Kuhlmann, Defect, in: Commutative Algebra - Noetherian and non-Noetherian perspectives, edited by M. Fontana, S.-E. Kabbaj, B. Olberding, and I. Swanson (Springer-Verlag, New York, 2011), 277-318.
[13] F.-V. Kuhlmann, The algebra and model theory of tame valued fields, to appear in J. reine angew. Math. (see http://www.degruyter.com/view/j/crelle.ahead-of-print/crelle-2014-0029/crelle-2014-0029.xml?format=INT)
[14] F.-V. Kuhlmann, Valuation Theory, book in preparation. Preliminary versions of several chapters are available on the web site: http://math.usask.ca/˜fvk/Fvkbook.htm.
[15] F.-V. Kuhlmann, M. Pank, and P. Roquette, Immediate and purely wild extensions of valued fields, Manuscripta math. **55**, 39–67 (1986).
[16] M. Nagata, Local rings (Wiley Interscience, New York, 1962).
[17] P. Ribenboim, Théorie des valuations (Les Presses de l'Université de Montréal, Montréal, 1st ed. 1964, 2nd ed. 1968).
[18] P. Vamos, Decomposition problems for modules over valuation domains, J. London Math. Soc. **41**, 10–26 (1990).
[19] S. Warner, Nonuniqueness of immediate maximal extensions of a valuation, Math. Scand. **56**, 191–202 (1985).
[20] S. Warner, Topological fields, Mathematics studies **157** (North Holland, Amsterdam, 1989).
[21] O. Zariski and P. Samuel, Commutative Algebra, Vol. II (Springer, New York–Heidelberg–Berlin, 1960).
[22] G. Whaples, Galois cohomology of additive polynomial and n-th power mappings of fields, Duke Math. J. **24**, 143–150 (1957).